Transformations of Graphs

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Hi. We're going to learn some ideas that will really help you when graphing any equation. There are several rules for transforming graphs of basic equations, and regardless of the type of equation it is, these rules always work! Once you have a good understanding of how to transform common graphs, you will be able to graph most equations faster than a graphing calculator can!

We'll begin by shifting graphs. Shifts are done by adding or subtracting values in the equation. But since shifts can move a graph left or right (horizontally) OR up and down (vertically) we need to pay attention on the clues that tell us **how** to move the basic graph.

Horizontal shifts occur when a value is added to (or subtracted from) the input value **before** anything else is done to the input. I call this before "processing" where the "process" is whatever the computation is that the function instructs us to do. The function notation $f(x+c)$ indicates that the value "c" is being added to the "x input" right away.

Here are two examples. In function $f(x+c)$, we are adding a value to "x" before it is squared. In this function, squaring is the "process." In function $g(x+c)$, we are adding a value to the "x" before the absolute value is found. Absolute value is the "processing" specified in function g. Horizontal movement can be to the right or to the left. In order to know which direction the graph will move we look at the operation. **Adding** a positive value before "processing" will move the graph to the **left**. <u>Subtracting</u> a value before "processing" will move the graph to the <u>right</u>. I know it seems backwards, but if you imagine moving across a graph from the left to the right, you can see that when you add a value, you would encounter that result sooner, so it will be to the left of where it would "normally" be.

In our example of $g(x)=|x|$ adding "2" before processing the absolute value will move the graph two units to the left. When a "3" is subtracted from the "x" before it is squared in function $f(x)=x^2$, the graph will move three units to the right of the original graph.

Shifts can also be made in a vertical direction. Shifts are always a result of adding or subtracting, so we need to discern when a graph will move vertically instead of horizontally. When a value is added or subtracted **after** the "processing" the graph will move **vertically**. That is, if the addition or subtraction is the last operation done in the function, you will see a vertical shift. The notation $f(x)+c$ indicates that the addition is done after the function has done processed the input.

Vertical shifts seem a bit more "natural" than the horizontal ones do. **Adding** a positive value after "processing" will move the graph **up**. <u>Subtracting</u> a value after "processing" will move the graph <u>downward</u>. In our example of $g(x)=|x|$, when we add "2" after the absolute value of the input is found, the graph will move up two units. When "3" is subtracted after the processing in function $f(x)=x^2$ (that is after squaring the input), the graph will move down three units from the original graph.

When you see a value added or subtracted to an equation, you know it will cause the graph to shift location. In order to know which direction the graph will shift, look to see **where** the value is added or subtracted. Also look for the **sign** of the value: see if the value is positive (added) or negative (subtracted).

When a graph flips over, we say it is "**reflected**." This happens when there is a factor of "–1" in the equation. You might not see the "1" in the equation. You may just see the negative sign. Keep in mind that will mean to multiply by "–1." A graph can flip (reflect) over the *y*-axis or the *x*-axis. The direction of the reflection will depend on **where** the "–1" is multiplied. If the input is multiplied by "–1" **right away**, the graph will reflect over the ***y*-axis**. If the "–1" is multiplied after the function "processes" the input, the graph will flip over the *x*-axis.

Seeing a "*–x*" under the radical might seem startling, but let's forge ahead with the analysis. "*–x*" means to multiply the input by "–1" as soon as the input value is inserted into the function. That means we are multiplying **before** processing. That will flip the graph across the ***y*-axis**. Now, if we think about it, this actually makes sense. We were a little alarmed when we first looked at the function to see a "*–x*" under the radical. However, if we input a negative number to start with, then multiply that by "–1," it will result in a positive number under the square root. That is a very good thing! And, the blue graph we see after reflecting the original graph agrees with our analysis. We need to use negative numbers as input; that is, the domain of this function are only negative values. The blue graph uses negative "*x*-values" in the ordered pairs.

We will need to look carefully at this example, too. At what point in the processing do we apply the negative in the function $f(x) = -x^2$? Remember that unless there are parentheses to group the negative with the "x", the negative is applied to the answer after the squaring is done. This function indicates a multiplication by "–1" after "processing." The graph will be reflected over the x-axis.

Graphs can also be **stretched** taller or flattened shorter by changing the equations, too. When a factor other than "1" is multiplied in the equation we will see a change in the shape of the graph. We're going to discuss these factors as if they are all positive. If they are negative, we will think of them as a "–1" times the absolute value of the factor. That will help us distinguish the stretch from the reflection. The graph of $f(x)=1/2x^2$ will be half as tall as the graph of $f(x)=x^2$. This shrinks the graph. The graph of $g(x)=3\sqrt{x}$ will be three times as tall as the graph of $g(x)=\sqrt{x}$. This stretches the graph taller.

To summarize, if we multiply by a value greater than "1," we will stretch the graph taller. If we multiply by a fractional value between "0" and "1," we will shrink the graph flatter. And yes, there is a concern about before "processing" and after "processing," but you don't need to worry about that until your Precalculus class. The graph of *g(x)* stretches when multiplied by "3." The graph of *f(x)* shrinks when multiplied by ½.

Of course, more than one operation can be done to a function. That will result in more than one transformation to the graph. When there are several transformations to make, it **does** make a difference what order you do them in. As you follow the Order of Operation to process the function's input, do the transformation in the same order. That is, do the transformations when you encounter the operation in the function that indicates the transformation. This means that any "before processing" operations will indicate the first transformations to be done to the graph. The transformations that result from multiplication will be done next. Then the "after processing" addition or subtraction will indicate the final transformation.

The "processing" of the function $f(x)=x^2$ is to square the input. There are three additional operations in the modified function that will indicate three transformations to the graph of $f(x)=x^2$. If you substituted an input value into the function for "*x*," the first operation you would do is to subtract "2" from the input. So the first transformation is to move the graph two units to the right. After subtracting, the input is "processed" by the function by squaring it. By the way, this process is what accounts for the shape of the graph in the first place. After "processing" we multiply by "½," so our graph will be half as tall as usual. The last operation is to subtract "1," so our graph will move one unit down.

Function $g(x)=\sqrt{x}$ "processes" an input by taking the square root. However, before the "processing" is done, the input is multiplied by a "–1." That will reflect the graph across the y-axis. After the "process" of finding the square root, we multiply by "3." This will stretch the graph three times taller than the original. The last operation is to add "1," so the graph will then move up one unit.

When there are two operations done to the input before "processing" we need to factor the coefficient of x in order to write the quantity as multiplication of a sum or difference. If we don't do this, we end up with the wrong graph.

When we factor the "–1" from "–x+2," we see the sign changes. What was an addition of "2" actually is a subtraction of "2." When we're ready for it, the horizontal shift will be to the right! However, the first thing we will do is respond to the multiplication by "–1." This is done before the "processing" of taking the square root, so we will reflect the graph across the y-axis. After that we are ready for the horizontal shift to the right.

You probably won't see this kind of combination very often, which is what makes this challenging to remember. But whenever there are two operations done before processing, factor first. If you forget to do that, your graph will be in the wrong location. Try these for a little more practice. Pause this lecture while you work on these two problems. Then we'll see how you did!

There are two places function *f(x)* received input. Since the quantity "*x*–1" was substituted into both of them, we are indicating that the very first operation to be done when inputting values is to subtract "1." That means the graph will move one unit to the right.

The groupings indicate that the "3" is to be multiplied to all of the processing. To determine the effect on the graph, first distribute the "3." The factor of "3" will make the graph three times taller. After we distribute, we see that the vertical shift is actually six units. It's vertical because this is the last operation done when evaluating the function.