Introduction to Exponential Functions

You will need to remember the Rules for Exponents (from the Review Resources Repository) as you study your reading. Remember how to simplify expressions with fractional exponents and negative exponents. Review the rules!

We're ready to work with Exponential Functions. The main difference between an exponential function and a polynomial (or algebraic) function is the location of the variable.

In a polynomial, the **variable** is the base and a **constant** is the exponent. $p(x) = x^3$

In an exponential function, the **variable** is the exponent and a **constant** is the base. $f(x) = 3^x$.

Here are a few details to notice.

1. We **don't use zero for the constant base** of an exponential function. If we did and tried $j(x)=0^x$, then what would happen if we tried to evaluate $j(-2)$? We would get $^1/_0$ which is undefined! We can't do that!

2. We use **only positive numbers for the constant base**. If we didn't restrict that, then we might write $k(x)=(-4)^x$. What would happen when we try to evaluate $k(^1/_2)$? This would give us $\sqrt{-4}$ which would be an imaginary result. We're going to stay away from these, too.

3. Also, we **don't use '1' for the base**. If we try $m(x)=1^x$, then what is the range? No matter what the x-value we use, the y-value is always going to be '1.' That means this function $m(x)$ would reduce to $m(x)=1$ which is a constant polynomial function.
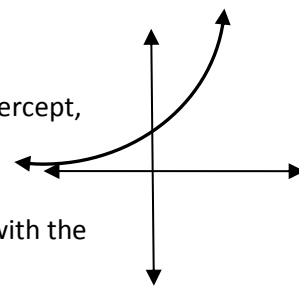
Here is an example: Given: $h(x)=3^x$,

       then $h(2)=3^2=9$,

       and $h(-1)=3^{-1}=^1/_3$

And if you're told that $h(x)=27$, that asks $3^x=27$, so would mean that $x=3$.

Graphs of Exponential Functions

All exponential graphs -- $f(x)=a^x$ -- have the same y-intercept. Because to find the y-intercept, we use x=0 and $f(0)=a^0=1$. So (0,1) is the common y-intercept no matter what the base of the exponential function is. Notice the graphs in section 5.3 of your textbook. The only difference in them is the 'sharpness' of the curve. We'll just work with the basic graph that looks something like this one.

The **domain** of exponentials is $(-\infty, +\infty)$; in other words all Real numbers can serve as exponents. The **range** of the "basic" exponential function is $(0, +\infty)$. The range is the output. And since we are only allowed to use positive number bases, we can only end up with positive results. There is no exponent that can turn a positive base into a zero nor into a negative result!! Thus, there are no x-intercepts on the graph.

Graphs of exponential functions can be transformed in the same way the polynomial graphs are.
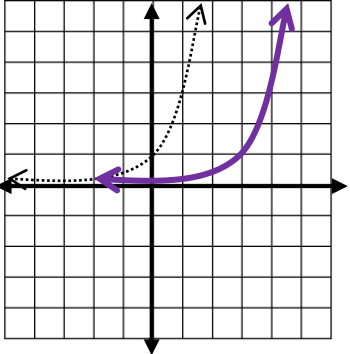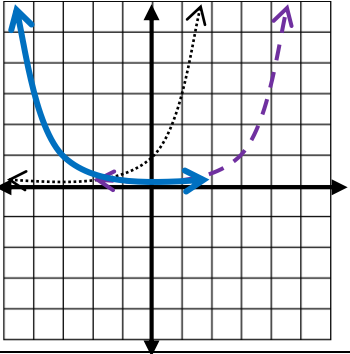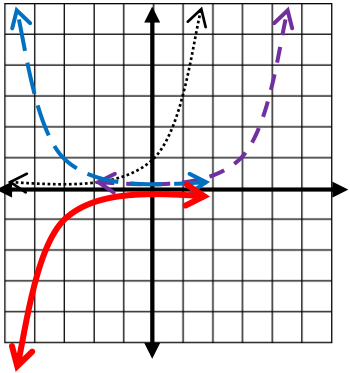
Vertical shifts: up=add after processing ; down= subtract after processing

Horizontal shifts: left=add before processing ; right=subtract before processing

Flips: over x-axis=multiply by –1 after processing; over y-axis=multiply by –1 before processing

Stretches: taller=multiply by value greater than 1; wider=multiply by value between 0 & 1

Here's an example of using the transformation rules from Activity 3.5 in Learning Unit 3, to graph an exponential function.

Graph: $f(x) = -2^{3-x}$.	
Think of 3–x as –x+3, so: $f(x) = -2^{-x+3}$ Factor the negative from the two terms in the exponent: $f(x) = -2^{-(x-3)}$ The graph will move **three units to the right** since three is subtracted before processing.	
The graph will **reflect across the y-axis** since we are multiplying by negative one before processing.	
Careful, now. What is the base of the exponent? It can't be –2, since negative numbers are not allowed as bases of exponential function. The base of the exponent is 2. After the processing is done, the result is multiplied by –1. This will reflect the graph **across the x-axis**.	
By the way, **IF** the base of the exponent were –2, it would have to be written with parenthesis: $(-2)^x$. This, however, is the situation that we can**NOT** do.	

Special Exponential Functions

There are two special exponential functions we commonly use.

1. Because our number system is **based on 10**, one useful exponential function is $t(x)=C10^x$.
2. Another very useful exponential function has a **base of "*e*."** *e* is NOT a variable. It is a number which occurs in nature (like π). It is not a number any person thought up, rather it is a number scientists discovered as they studied growth and decay in the "natural" world. *e* is an irrational number approximately equal to 2.7182818.... (never–ending, non–repeating). Thus $n(x)=Ce^x$ is called the "natural" exponential function.

Exponential Growth and Decay

The general exponential equation is $f(x)=a^x$ or $f(x)=Ca^x$ where the *C* indicates there is some initial amount that will be increased or decreased by the repeated factor *a*. There are two ways to tell if an exponential function is describing "growth" or "decay." If the base of the exponent is a fraction, the initial amount will decrease. That is if *0<a<1*, the equation describes "decay" of the initial amount.

> So, the function $g(x)=4^x$ describes growth because the base is greater than 1.
> The function $h(x) = (¼)^x$ describes decay because the base is less than 1.
>
> Remember there is another way to write ¼. We can also write it as 4^{-1}, so h(x) could also be written: $h(x) = 4^{-x}$.
>
> In either format, *h(x)* would give the same outputs and have the same graph.

Thus, the second way we can identify whether an exponential function describes growth or decay is to look at the exponent. If the base is greater than 1, but the exponent is negative, it will still describe a "decay" of the initial amount.

Some Applications Explained:

$A = P(1 + {}^r/_n)^{nt}$	Compound Interest Formula. "n" is the number of times the interest is compounded in a year (i.e. quarterly means n=4). "r" is the interest rate expressed in decimal form. "t" is the number of years the account is active. "P" is the principal, the amount you begin with. "A" is the amount in the account at the end of the term. It includes the original principal and the interest.
$A = Pe^{rt}$	Interest Compounded Continuously. When the rate is positive, the account will grow. (That's the way investors hope their accounts will work!)
$y = Ca^x$	Exponential Growth. As long as the exponent is positive and the base is greater than 1, this formula expresses growth. The "C" is the initial amount of material present (beginning population, or beginning radioactivity, etc.)
$y = C({}^1/_2)^{x/k}$	Half Life. When the base of the exponent is ½, half of the substance decays each year. "x" represents the number of years and "k" is the number of years until half of the substance has decayed. So if x=k, the exponent is "1" and half of the substance is gone. If x<k, then not enough time has passed for the substance to be half gone. (i.e. If x=20 and k=60, then ${}^x/_k = {}^1/_3$ in the exponent. So only ${}^1/_3$ of the time has passed that would be necessary for half of the substance to decay.